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On Discretization Errors in Nonlinear Approximation Problems

REMBERT REEMTSEN

Fachbereich Mathematik, Technische Universität Berlin, 1000 Berlin 12, West Germany

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In this paper we extend the results on discretization errors in linear L_∞-approximation problems by Cheney [4] to a broad class of nonlinear L_p-approximation problems with constraints where $1 \leqslant p \leqslant \infty$. © 1987 Academic Press, Inc.

1. INTRODUCTION

Let $(X, \|\cdot\|_X)$ be a linear normed space of dimension n generated by $v_1,..., v_n$ and let $A \subseteq X$ be a nonempty closed set. Further, let $1 \leqslant p \leqslant \infty$ and $B \subset \mathbb{R}^s$ be a compact set; in particular, let $B = [c, d]$ if $1 \leqslant p < \infty$. Then $C(B)$ shall be the space of all real-valued continuous functions on B equipped with the L_p-norm

$$\|f\|_B = \begin{cases} \max\limits_{\xi \in B} |f(\xi)| & \text{if } p = \infty, \\ \left\{\int_c^d |f(\xi)|^p \, d\xi\right\}^{1/p} & \text{if } 1 \leqslant p < \infty, \end{cases} \tag{1}$$

for $f \in C(B)$. Finally, let $r \in C(B)$ be fixed and $T: A \to C(B)$ be a continuous operator. Then we consider the nonlinear approximation problem

$$(P) \quad \text{Minimize } \|r - Ta\|_B \text{ on } A$$

with

$$\rho = \inf\{\|r - Ta\|_B \,|\, a \in A\}. \tag{2}$$

We say that $\hat{a} \in A$ is a solution of (P) if the infimum in (2) is achieved for $\hat{a} \in A$.

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In practice it usually is impossible to solve (P) directly. In order to handle (P) numerically, one will, therefore, try to "simplify" the problem through discretization in the following sense. Let $B_k \subseteq B$ be compact; in particular, for $1 \leqslant p < \infty$ let B_k consist of k points $\xi_i^{(k)} \in \mathbb{R}$, $1 \leqslant i \leqslant k$, with

$$c \leqslant \xi_1^{(k)} < \xi_2^{(k)} < \cdots < \xi_k^{(k)} \leqslant d$$

where $\xi_i^{(k)} \in [y_{i-1}^{(k)}, y_i^{(k)}]$ and the $y_i^{(k)}$ are such that $y_{i-1}^{(k)} < y_i^{(k)}$ and

$$[c, d] = [y_0^{(k)}, y_1^{(k)}] \cup [y_1^{(k)}, y_2^{(k)}] \cup \cdots \cup [y_{k-1}^{(k)}, y_k^{(k)}].$$

Setting

$$h_i^{(k)} = y_i^{(k)} - y_{i-1}^{(k)}, \quad 1 \leqslant i \leqslant k,$$

we define the discrete semi-norm corresponding to (1) by

$$\|f\|_{B_k} = \begin{cases} \max\limits_{\xi \in B_k} |f(\xi)| & \text{if } p = \infty, \\ \left\{ \sum\limits_{i=1}^{k} |f(\xi_i^{(k)})|^p \, h_i^{(k)} \right\}^{1/p} & \text{if } 1 \leqslant p < \infty. \end{cases}$$

We assume further $A_k \supseteq A$ to be a closed upper set of A in X and $T: A_k \to C(B)$ to be defined and continuous on A_k. (Note that the range of T is in $C(B) \subseteq C(B_k)$). If e.g. A is the solution set of infinitely many linear constraints, A_k may be the solution set of finitely many of them. Then instead of (P) one will try to solve the problem

$$(P_k) \quad \text{Minimize } \|r - Ta\|_{B_k} \text{ on } A_k.$$

In correspondence with (2) we set

$$\rho_k = \inf\{ \|r - Ta\|_{B_k} \mid a \in A_k \}$$

and call $\hat{a}_k \in A_k$ a solution of (P_k) if it exists.

A question of obvious interest is now: provided that $\{B_k\}_{k \in \mathbb{N}}$ and $\{A_k\}_{k \in \mathbb{N}}$ are sequences which in an appropriate way converge to B and A, resp., do the ρ_k tend to ρ for $k \to \infty$? It is well known that this question can be answered in the affirmative, if T is linear and e.g. $A = A_k = X$ (cf. [4, 20]). Furthermore, it has been shown that there are nonlinear problems where the ρ_k do not converge to ρ (see [6] and the examples below). This is disturbing in view of the fact that almost all algorithms for the solution of approximation problems, including those for the solution of semi-infinite programming problems (cf. [9]), are based on discretization. Hence it is of big practical importance to identify such operators T for which the requested convergence can be shown.

For linear unconstrained problems, i.e., if T is linear and $A = A_k = X$, the problem of the convergence of discretization errors for $p = \infty$ has been studied first by Cheney (see [4]). The author of [20] gives a corresponding proof for $p = 1$ which can be easily transferred to the case $1 < p < \infty$. Results on the rate of convergence have been obtained in [3] for $p = \infty$ and in [14] for $p = 1$. Finally, for linear problems which are equivalent to certain semi-infinite programming problems a convergence result of the requested type can be found in [9].

Concerning discrete nonlinear approximations, almost exclusively exponential and rational approximation problems have been investigated so far (cf. [1, 2, 11, 15, 17, 18, 19]). However, it has been noted that the discretization of approximation problems can be considered as a special perturbation problem in optimization and hence can be tackled with the available theories in this connection (see e.g. [10]). We shall discuss the relation of our results to this latter approach in Remark 1 below. Our aim here is to derive sufficient conditions for the convergence of the ρ_k to ρ which can be actually verified in many circumstances.

The plan of this paper is as follows:

In Sect. 2 we prove convergence of the ρ_k under the assumptions that the (P_k), $k \geqslant \hat{k}$, possess solutions $\hat{a}_k \in A_k$ and that $\{T\hat{a}_k\}_{k \geqslant \hat{k}}$ is equicontinuous on B. In the remaining sections we are concerned with the verification of these assumptions in specific situations. So we first show in Sect. 3 that in case T is linear existence of the $\hat{a}_k$ (which often can be guaranteed then) implies the equicontinuity of $\{T\hat{a}_k\}_{k \geqslant \hat{k}}$. Thereby results in [4] and [20] are generalized since we allow here problems with constraints and since the proofs are valid for every L_p-norm. Then in Sect. 4 we turn to nonlinear problems. We first provide a lemma which for many nonlinear L_∞-problems ensures the requested existence of solutions and equicontinuity. Afterwards we study the corresponding properties of T for $1 \leqslant p < \infty$.

We have applied our results successfully to a variety of nonlinear approximation problems, in particular to problems where T has been a differential operator. In the final part of this paper we present two such applications and thereby show that some former results of other authors follow quite easily from our theory.

2. A Convergence Theorem

We begin by providing some definitions. Let T be defined on $S \subseteq X$. Then for $\alpha \geqslant 0$ we set

$$C_\alpha(S) = \{a \in S \mid \|Ta\|_B \leqslant \alpha\} \tag{3}$$

and correspondingly

$$C_\alpha^k(S)=\{a\in S\mid \|Ta\|_{B_k}\leqslant\alpha\}. \tag{4}$$

Further, we fix some $a_0\in A$ and a number $\sigma>0$ ($\sigma=0$ is possible if $p=\infty$) and define

$$\alpha_0=\|r-Ta_0\|_B+\|r\|_B+\sigma \tag{5}$$

Moreover, if M and Q are nonempty subsets of a linear normed space $(Z, \|\cdot\|_Z)$, we write

$$h(M, Q)=\sup_{x\in Q}\inf_{y\in M}\|x-y\|_Z.$$

If $\{M_k\}_{k\in\mathbb{N}}$ is a sequence of nonempty subsets of Z, then $\lim_{k\to\infty} h(M, M_k)=0$ if and only if for each $\varepsilon>0$ there is a number $k(\varepsilon)\in\mathbb{N}$ such that for all $k\geqslant k(\varepsilon)$

$$M_k\subseteq M_\varepsilon=\{x\in Z\mid \inf_{y\in M}\|x-y\|_Z\leqslant\varepsilon\}.$$

We are now in the position to state the following two assumptions.

ASSUMPTION 1. $\{B_k\}_{k\in\mathbb{N}}$ is a sequence of compact subsets of B in $(\mathbb{R}^N, \|\cdot\|_2)$ with $\lim_{k\to\infty} h(B_k, B)=0$ where $\|\cdot\|_2$ is the Euclidean norm.

ASSUMPTION 2. $\{A_k\}_{k\in\mathbb{N}}$ is a sequence of closed subsets of $(X, \|\cdot\|_X)$ where

(i) $A\subseteq\cdots\subseteq A_{k+1}\subseteq A_k\subseteq\cdots\subseteq A_1\subseteq X$.

Further, T is defined on $A_{\hat{k}}$ for a $\hat{k}\in\mathbb{N}$, i.e. $T\colon A_{\hat{k}}\to C(B)$, and either

(ii) $\lim_{k\to\infty} h(A, A_k)=0$ or

(ii)′ $\lim_{k\to\infty} h(C_{\alpha_0}(A), C_{\alpha_0}^k(A_k))=0$.

With regard to Assumption 2 let us mention that in applications (ii)′ may be satisfied because of the possible boundedness of the sets $C_{\alpha_0}(A)$ and $C_{\alpha_0}^k(A_k)$ while at the same time (ii) may not be true. Obviously, Assumption 2 is fulfilled if $A=A_k$ for all $k\in\mathbb{N}$.

The level sets (3) and (4) play an important role in approximation theory in connection with questions of existence of solutions and of convergence of algorithms (e.g. [16]) as well as questions arising with discretization (see Sect. 4). This is due to their possible boundedness and the following facts.

LEMMA 1. (i) $\rho = \inf\{\|r - Ta\|_B | a \in C_{\alpha_0}(A)\}$.
If $\hat{a} \in A$ *is a solution of* (P), $\hat{a}$ *is element of* $C_{\alpha_0}(A)$.

(ii) *Under Assumptions* 1 *and* 2 *there is a number* $k_0 \geqslant \hat{k}$, *where* $k_0 = \hat{k}$ *if* $p = \infty$, *such that for all* $k \geqslant k_0$

$$\rho_k = \inf\{\|r - Ta\|_{B_k} | a \in C^k_{\alpha_0}(A_k)\}.$$

Further, if $\hat{a}_k \in A_k$ *solves* (P_k), $k \geqslant k_0$, $\hat{a}_k$ *is in* $C^k_{\alpha_0}(A_k)$.

Proof. (i)

$$\begin{aligned}\rho &= \inf\{\|r - Ta\|_B | a \in A \quad \text{and} \quad \|r - Ta\|_B \leqslant \|r - Ta_0\|_B\} \\ &\geqslant \inf\{\|r - Ta\|_B | a \in A \quad \text{and} \quad \|Ta\|_B \leqslant \alpha_0\} \geqslant \rho.\end{aligned}$$

From the same inequalities it is obvious that a solution $\hat{a} \in A$ of (P) is in $C_{\alpha_0}(A)$ if it exists.

(ii) By virtue of Assumption 2, a_0 is element of A_k. Hence

$$\rho_k \geqslant \inf\{\|r - Ta\|_{B_k} | a \in A_k \quad \text{and} \quad \|Ta\|_{B_k} \leqslant \|r - Ta_0\|_{B_k} + \|r\|_{B_k}\}.$$

If $p = \infty$,

$$\|r - Ta_0\|_{B_k} + \|r\|_{B_k} \leqslant \alpha_0 \tag{6}$$

for all $k \geqslant \hat{k}$ is obvious. In case $1 \leqslant p < \infty$, (6) is true for all sufficiently large k by virtue of the definition of the Riemann integral. The remainder of the proof follows the proof of (i).

We now give the main result of this paper. For that we note that $N = \bigcup_{k \geqslant \hat{k}} C^k_{\alpha_0}(A_k)$ is a subset of $A_{\hat{k}}$ and define

$$N_\varepsilon = \{a \in X \mid \|a - b\|_X \leqslant \varepsilon \text{ for a } b \in N\} \tag{7}$$

to be an ε-neighborhood of N in X.

THEOREM 1. *Let Assumptions* 1 *and* 2 *be fulfilled. Further, let* T *be continuous on* $A_{\hat{k}}$ *for* $\hat{k} \in \mathbb{N}$. *Moreover, let* T *be uniformly continuous on* $N_{\hat{\varepsilon}} \cap A_{\hat{k}}$ *for an* $\hat{\varepsilon} > 0$ *in case* $A \neq A_k$ *for some* $k \geqslant \hat{k}$. *Further, let* (P_k) *have a solution* $\hat{a}_k \in A_k$ *for each* $k \geqslant \hat{k}$ *and let* $\{T\hat{a}_k\}_{k \geqslant \hat{k}}$ *be equicontinuous on B. Then we have*:

(i) $\lim_{k \to \infty} \rho_k = \rho$.

(ii) $\lim_{k \to \infty} \|r - T\hat{a}_k\|_B = \rho$ ($\hat{a}_k$ *may not be in* A).

If in addition there is a constant C *so that* $\|\hat{a}_k\|_X \leqslant C$ *for all* $k \geqslant \hat{k}$, *then*

(iii) $\{\hat{a}_k\}_{k \geqslant \hat{k}}$ *possesses at least one accumulation point which lies in* A *and each such accumulation point solves* (P). *Moreover, if* (P) *has a unique solution* $\hat{a} \in A$, *then* $\lim_{k \to \infty} \|\hat{a} - \hat{a}_k\|_X = 0$.

Proof. (i) With respect to Assumption 2 we define either

$$M = A \qquad \text{and} \qquad M_k = A_k$$

or

$$M = C_{x_0}(A) \qquad \text{and} \qquad M_k = C^k_{x_0}(A_k).$$

In any case M and M_k, $k \geqslant \hat{k}$, are nonempty sets. We fix now $\varepsilon \in (0, \hat{\varepsilon}]$ and set

$$\Delta(\varepsilon) = \begin{cases} 0, & \text{if } A = A_k \quad \text{for all } k \geqslant \hat{k}, \\ \sup\{ \|Ta - Tb\|_B \mid \|a - b\|_X \leqslant \varepsilon,\ a, b \in N_{\hat{\varepsilon}} \cap A_{\hat{k}} \}, & \text{else.} \end{cases}$$

Obviously, we have $\Delta(\varepsilon) \to 0$ for $\varepsilon \to 0$.

Now we observe first that by Lemma 1 there is a number $k_0 \geqslant \hat{k}$ such that $\hat{a}_k$ is in $C^k_{x_0}(A_k) \subseteq M_k$ if $k \geqslant k_0$. Further, due to Assumption 2, there exists a number $k_1 \geqslant \hat{k}$ such that we have $M_k \subseteq M_\varepsilon$ for all $k \geqslant k_1$. Hence if $k \geqslant \max(k_0, k_1)$, for $\hat{a}_k$ we can find an element $\tilde{a}_k \in M$ with $\|\tilde{a}_k - \hat{a}_k\|_X \leqslant \varepsilon$. In particular, we can choose $\tilde{a}_k = \hat{a}_k$ if $A = A_k$ for all $k \geqslant \hat{k}$.

Next we note that in any case $M \subseteq A \subseteq A_{\hat{k}}$ holds true. Therefore we can infer that $\hat{a}_k$ as well as $\tilde{a}_k$ belong to $(N_\varepsilon \cap A_{\hat{k}}) \subseteq (N_{\hat{\varepsilon}} \cap A_{\hat{k}})$ for all $k \geqslant \max(k_0, k_1)$. Thus if $k \geqslant \max(k_0, k_1)$ we have

$$\inf_{a \in A} \|r - Ta\|_B - \|r - T\hat{a}_k\|_B \leqslant \|r - T\tilde{a}_k\|_B - \|r - T\hat{a}_k\|_B \leqslant \Delta(\varepsilon). \tag{8}$$

Our next objective now will be to study the expression

$$\|r - T\hat{a}_k\|_B - \inf_{a \in A_k} \|r - Ta\|_{B_k} = \|r - T\hat{a}_k\|_B - \|r - T\hat{a}_k\|_{B_k}. \tag{9}$$

For $f \in C(B)$ we define

$$\omega(f, \varepsilon) = \sup\{ |f(x) - f(y)| \mid \|x - y\|_2 \leqslant \varepsilon, \qquad x, y \in B\} \tag{10}$$

to be the modulus of continuity of f on B. Then we set

$$\mu(\varepsilon) = \sup_{k \geqslant \hat{k}} \omega(T\hat{a}_k, \varepsilon)$$

and we observe that $\mu(\varepsilon)$ tends to zero with $\varepsilon \to 0$ because of the equicontinuity of $\{T\hat{a}_k\}_{k \geqslant \hat{k}}$ on B. Further, due to Assumption 1 there is a number

$k_2 \geqslant \hat{k}$ so that $h(B_k, B) \leqslant \varepsilon$ for all $k \geqslant k_2$. Let now $k \geqslant k_2$ and let first $1 \leqslant p < \infty$. Then choose $\theta_i^{(k)} \in B$ such that

$$|r(\theta_i^{(k)}) - (T\hat{a}_k)(\theta_i^{(k)})| = \max_{y_{i-1}^{(k)} \leqslant x \leqslant y_i^{(k)}} |r(x) - (T\hat{a}_k)(x)|.$$

Hence, we have $\|\xi_i^{(k)} - \theta_i^{(k)}\|_2 \leqslant \varepsilon$ for all $i \in \{1,..., k\}$. If we use now

$$\|r - T\hat{a}_k\|_B^p = \sum_{i=1}^k \int_{y_{i-1}^{(k)}}^{y_i^{(k)}} |r(x) - (T\hat{a}_k)(x)|^p \, dx$$
$$\leqslant \sum_{i=1}^k |r(\theta_i^{(k)}) - (T\hat{a}_k)(\theta_i^{(k)})|^p \, h_i^{(k)}$$

and apply Minkowski's inequality twice, we obtain

$$\begin{aligned}
&\|r - T\hat{a}_k\|_B - \|r - T\hat{a}_k\|_{B_k} \\
&\quad \leqslant \left\{ \sum_{i=1}^k |[r(\theta_i^{(k)}) - (T\hat{a}_k)(\theta_i^{(k)})] - [r(\xi_i^{(k)}) - (T\hat{a}_k)(\xi_i^{(k)})]|^p \, h_i^{(k)} \right\}^{1/p} \\
&\quad \leqslant \left\{ \sum_{i=1}^k |r(\theta_i^{(k)}) - r(\xi_i^{(k)})|^p \, h_i^{(k)} \right\}^{1/p} \\
&\qquad + \left\{ \sum_{i=1}^k |(T\hat{a}_k)(\theta_i^{(k)}) - (T\hat{a}_k)(\xi_i^{(k)})|^p \, h_i^{(k)} \right\}^{1/p} \\
&\quad \leqslant K\{\omega(r, \varepsilon) + \mu(\varepsilon)\} \qquad (11)
\end{aligned}$$

where $K = (d-c)^{1/p}$. In case $p = \infty$ (11) is easily seen to be true with $K = 1$. Hence combining (8), (9), and (11) we get for $1 \leqslant p \leqslant \infty$ and $k \geqslant \max(k_0, k_1, k_2)$

$$\inf_{a \in A} \|r - Ta\|_B - \inf_{a \in A_k} \|r - Ta\|_{B_k} \leqslant \Delta(\varepsilon) + K\{\omega(r, \varepsilon) + \mu(\varepsilon)\}. \qquad (12)$$

By virtue of our assumptions, the right-hand side of (12) tends to zero with $\varepsilon \to 0$. For $p = \infty$ it is easily seen that the left-hand side of (12) is always nonnegative since we have $B_k \subseteq B$ and $A \subseteq A_k$. Thus, (i) is proved in this case. If $1 \leqslant p < \infty$, for every $a \in A$ there is a number $E(a, h^{(k)})$ for $h^{(k)} = \max_{1 \leqslant i \leqslant k} h_i^{(k)}$ so that $E(a, h^{(k)}) \to 0$ for $h^{(k)} \to 0$ and

$$\left\{ \int_c^d |r(x) - (Ta)(x)|^p \, dx \right\}^{1/p}$$
$$= \left\{ \sum_{i=1}^k |r(\xi_i^{(k)}) - (Ta)(\xi_i^{(k)})|^p \, h_i^{(k)} \right\}^{1/p} + E(a, h^{(k)}).$$

We note that $h^{(k)}$ tends to zero if and only if $h(B_k, B)$ does so. Now we choose $b \in A$ so that $\|r - Tb\|_B - \inf_{a \in A} \|r - Ta\|_B < \varepsilon$. Then there exists a $k_3 \geqslant \hat{k}$ such that $|E(b, h^{(k)})| \leqslant \varepsilon$ for all $k \geqslant k_3$. Hence

$$\rho + \varepsilon \geqslant \|r - Tb\|_{B_k} + E(b, h^{(k)}) \geqslant \rho_k - \varepsilon, \quad k \geqslant k_3. \tag{13}$$

Combination of (12) and (13) yields the requested result.

(ii) From (8) we have

$$\rho - \rho_k - \Delta(\varepsilon) \leqslant \|r - T\hat{a}_k\|_B - \|r - T\hat{a}_k\|_{B_k} \tag{14}$$

for all $k \geqslant \max(k_0, k_1)$ and from (11) and (14) we obtain recalling (i)

$$|\, \|r - T\hat{a}_k\|_B - \|r - T\hat{a}_k\|_{B_k}| \to 0 \text{ for } k \to \infty. \tag{15}$$

Finally, we write

$$|\rho - \|r - T\hat{a}_k\|_B| \leqslant |\rho - \rho_k| + |\, \|r - T\hat{a}_k\|_{B_k} - \|r - T\hat{a}_k\|_B|$$

so that (ii) follows from (i) and (15).

(iii) By assumption all $\hat{a}_k$, $k \geqslant \hat{k}$, are elements of the compact set $\{a \in X | \|a\|_X \leqslant C\}$. Hence there exists a subsequence $\{\hat{a}_{k_i}\}_{i \in \mathbb{N}}$ of $\{\hat{a}_k\}_{k \geqslant \hat{k}}$ which converges to an element $\bar{a} \in X$. Since again by Lemma 1 $\hat{a}_{k_i}$ is in M_{k_i} for all sufficiently large i and since M is a closed set, it follows easily from Assumption 2 that $\bar{a}$ lies in M. Further, since T is continuous on $A_{\hat{k}}$, $\|r - T\hat{a}_{k_i}\|_B$ tends to $\|r - T\bar{a}\|_B$ for $i \to \infty$. Therefore, from (ii) we get that $\bar{a}$ is a solution of (P). Moreover, if (P) possesses a unique solution $\hat{a} \in A$, every convergent subsequence of $\{\hat{a}_k\}_{k \geqslant \hat{k}}$ and thus the whole sequence converges to $\hat{a}$.

Remark 1. If (P) is being considered as a special optimization problem and (P_k) as a corresponding problem with perturbed data, the questions of this paper can be attacked with the perturbation theories in optimization. However, as we shall show, little is gained by such an approach since the verification of the assumptions of these theories for our problem requires most of the arguments of the proofs presented here.

Let us first relate to the results in [12]. We can fit the problems (P) and (P_k) into the model considered in [12] if we choose the parameters there as follows:

$$E := \bigcup_{k \geqslant \hat{k}} C^k_{\alpha_0}(A_k) \cup C_{\alpha_0}(A) \qquad \text{(equipped with } \|\cdot\|_X),$$

$$X := C_{\alpha_0}(A), \qquad S := X, \qquad f(x) := \|r - Tx\|_B,$$

$$X_k := C^k_{\alpha_0}(A_k), \qquad S_k := X_k, \qquad f_k(x) := \|r - Tx\|_{B_k},$$

where we suppose that Assumptions 1, 2(i), and (ii)$'$ are fulfilled and T is continuous on $A_{\hat{k}}$. (The reader may verify that in case of Assumption 2(ii)$'$ Theorem 1 and Lemma 3 remain valid for E instead of $N_{\hat{\varepsilon}} \cap A_{\hat{k}}$). Then Satz 3.1 in [12] shows first that for the proof of Theorem 1 (i) the assumptions of the existence of the $\hat{a}_k$ and the equicontinuity of $\{T\hat{a}_k\}_{k \geq \hat{k}}$ on B can be replaced by the "uniform equicontinuity of the f_k on E" and condition (3.1) ebd. However, if we wish to establish this equicontinuity of the f_k here for nonlinear T, we shall assume the compactness of E (which implies the existence of the $\hat{a}_k$ (cf. Lemmas 1 and 3)) and, moreover, have to verify condition (3.3) in [12] (see Lemma 3.2 ebd.). The proof of this latter condition can be accomplished with arguments similar to those which we need for the proof of Theorem 1 (i) above. The reader may further confirm that in case T is linear (where the assumption of the compactness of E usually is too strong) the proof of the assumptions of Satz 3.1 requires similar estimates as we shall derive them in the proof of Theorem 2 below.

Corresponding considerations hold true for the theorem in [10]. (We also note that condition (1.1) there can only be fulfilled if $p = \infty$). Furthermore, the author of [13] summarizes a large number of results in parametric optimization by using the concept of set-valued mappings. But again the verification of the assumptions of the respective theorems for our problem necessitates the same boundedness and equicontinuity assumptions which we referred to above (cf. condition (2.2) there).

3. Linear Operators

If T is a linear operator, the existence of solutions to (P) and (P_k) can often be proved with classical arguments. For example, by the following Lemma (P) as well as (P_k) has a solution if, e.g., $A = X$ and if T is the identical operator.

Lemma 2. *Let T be linear. If the image $T(A)$ of A under T is closed, (P) possesses a solution $\hat{a} \in A$. Correspondingly, if $T(A_k)$ $(k \geq \hat{k})$ is closed in $C(B_k)$ with respect to $\|\cdot\|_{B_k}$, (P_k) has a solution $\hat{a}_k \in A_k$.*

Proof. By our assumptions, the set

$$\{Ta \in T(A) \mid \|Ta\|_B \leq \alpha_0\} \subseteq T(X),$$

where α_0 is defined by (5), is compact. Hence the existence of $\hat{a} \in A$ can be concluded from Lemma 1 and Weierstrass' theorem. Correspondingly, the existence of $\hat{a}_k$ is proved.

Let us further mention that for $p = \infty$ another tool for the verification of the existence of solutions to (P) and (P_k) is given by Lemma 3. In combination with Theorem 1 and Lemma 2 the following theorem now extends results of [4, 20] for $p=\infty$ and $p=1$, respectively.

THEOREM 2. *Let Assumptions* 1 *and* 2 *be fulfilled and let T be a linear operator from X into $C(B)$. Further let (P_k) have a solution $\hat{a}_k \in A_k$ for each $k \geqslant \hat{k}$. Then the following is true*:

(i) $\{T\hat{a}_k\}_{k \geqslant \hat{k}}$ *is equicontinuous on B.*

(ii) *If $Tv_1, \ldots, Tv_n$ are linearly independent on B, there is a constant C so that $\|\hat{a}_k\|_X \leqslant C$ for all $k \geqslant \hat{k}$.*

Proof. (i) Let $w_i = Tv_i$ for $i \in \{1, \ldots, n\}$ and let the first $m \leqslant n$ of the w_i be linearly independent. Further, for $\varepsilon > 0$ we define

$$\Omega_m(\varepsilon) = \max_{1 \leqslant i \leqslant m} \omega(w_i, \varepsilon)$$

with ω (10). Obviously, $\Omega_m(\varepsilon)$ tends to zero if $\varepsilon \to 0$. Then if $T\hat{a}_k = \sum_{i=1}^m \beta_i^k w_i$ and $\|x-y\|_2 \leqslant \varepsilon$ we obtain

$$|(T\hat{a}_k)(x) - (T\hat{a}_k)(y)| \leqslant \|\beta^k\|_1 \, \Omega_m(\varepsilon)$$

where $\|\cdot\|_1$ is the l_1-norm in $\mathbb{R}^m$. Hence the proof of (i) is completed if there is a constant M such that $\|\beta^k\|_1 \leqslant M$ for all $k \geqslant \hat{k}$.

For that we define

$$\theta_m = \min_{\|\beta\|_1 = 1} \left\| \sum_{i=1}^m \beta_i w_i \right\|_B .$$

Due to the linear independence of the w_i we have $\theta_m > 0$.

Now we choose $\varepsilon > 0$ sufficiently small so that $\Omega_m(\varepsilon) \leqslant \theta_m/(2K)$ with K from (11). Then there is a number $k_1 \geqslant \hat{k}$ so that $h(B_k, B) \leqslant \varepsilon$ for all $k \geqslant k_1$. If we make use of (11) for $r=0$, we finally get for all $\beta \in \mathbb{R}^m$ and $k \geqslant k_1$

$$\begin{aligned} \theta_m \|\beta\|_1 &\leqslant \left\| \sum_{i=1}^m \beta_i w_i \right\|_B \leqslant K\,\Omega_m(\varepsilon)\|\beta\|_1 + \left\| \sum_{i=1}^m \beta_i w_i \right\|_{B_k} \\ &\leqslant \frac{\theta_m}{2} \|\beta\|_1 + \left\| \sum_{i=1}^m \beta_i w_i \right\|_{B_k} \end{aligned}$$

so that by Lemma 1 we have for all $k \geqslant \max(k_0, k_1)$

$$\|\beta^k\|_1 \leqslant \frac{2}{\theta_m} \left\| \sum_{i=1}^m \beta_i^k w_i \right\|_{B_k} \leqslant \frac{2\alpha_0}{\theta_m} \tag{16}$$

(ii) Let now $\hat{a}_k = \sum_{i=1}^n \gamma_i^k v_i$. Then in the proof of (i) we have $m=n$ and $\beta_i^k = \gamma_i^k$, $1 \leqslant i \leqslant n$, so that (ii) is a consequence of (16).

Remark 2. Let $A = X$, T be linear, and $Tv_1, \ldots, Tv_n$ be linearly independent. If $1 < p < \infty$, then (P) always possesses a unique solution. In case $p = 1$ or $p = \infty$, it is well known that a solution of (P) is unique if $Tv_1, \ldots, Tv_n$ form a so-called Haar system on B (e.g. [20]).

4. Nonlinear Operators

If T is a nonlinear operator, in general the existence of solutions to (P) and (P_k) is difficult to verify, not to speak of the uniqueness of solutions. In 4.1 we will provide a condition which for $p = \infty$ guarantees existence of solutions $\hat{a} \in A$ and $\hat{a}_k \in A_k$ and in many situations equicontinuity of $\{T\hat{a}_k\}_{k \in \mathbb{N}}$. In 4.2 we will study this condition for the case that $1 \leq p < \infty$. Finally, in 4.3 we will apply our results to two examples.

4.1. *The case $p = \infty$*

Throughout this subsection we assume $p = \infty$. Then we can state the following lemma.

Lemma 3. *Let Assumptions* 1 *and* 2 *be satisfied and let in addition*

$$B_k \subseteq B_{k+1} \subseteq \cdots \subseteq B, \quad k \in \mathbb{N}. \tag{17}$$

Further, let there exist a number $\hat{k}$ such that T is continuous on $A_{\hat{k}}$ and $C_{\alpha_0}^{\hat{k}}(A_{\hat{k}})$ is bounded. Then we have:

(i) *(P) and (P_k), $k \geq \hat{k}$, possess solutions $\hat{a} \in A$ and $\hat{a}_k \in A_k$, resp.*

(ii) *T is uniformly continuous on $N_\varepsilon \cap A_{\hat{k}}$ for each $\varepsilon > 0$ with N_ε (7).*

(iii) *There is a constant C such that $\|\hat{a}_k\|_X \leq C$ for all $k \geq \hat{k}$.*

Proof. It can be easily proved that due to our assumptions

$$C_{\alpha_0}(A) \subseteq \cdots \subseteq C_{\alpha_0}^{k+1}(A_{k+1}) \subseteq C_{\alpha_0}^{k}(A_k), \qquad k \in \mathbb{N}, \tag{18}$$

holds true. Hence $C_{\alpha_0}(A)$ and $C_{\alpha_0}^{k}(A_k)$, $k \geq \hat{k}$, are bounded and by the continuity of T also closed sets in X. So recalling Lemma 1 we can establish (i). Finally, (ii) and (iii) follow from the fact that (18) implies $\bigcup_{k \geq \hat{k}} C_{\alpha_0}^{k}(A_k) = C_{\alpha_0}^{\hat{k}}(A_{\hat{k}})$.

Statement (iii) of Lemma 3 often ensures the equicontinuity of the $\{T\hat{a}_k\}_{k \geq \hat{k}}$. Furthermore (18) shows that boundedness of $C_{\alpha_0}^{\hat{k}}(A_{\hat{k}})$ for a $\hat{k} \in \mathbb{N}$ guarantees boundedness of $C_{\alpha_0}(A)$. Unfortunately, in practice it is easier to examine $C_{\alpha_0}(A)$ than $C_{\alpha_0}^{\hat{k}}(A_{\hat{k}})$, and, as the following example shows, boundedness of $C_{\alpha_0}(A)$ by no means implies necessarily boundedness of any $C_{\alpha_0}^{k}(A_k)$. Moreover, this example shows that the uniform problem can have a solution while at the same time none of the discrete problems possesses one (see also [6] for another example).

EXAMPLE 1. Let $A = X = \mathbb{R}$, $B = [0, 1]$, $r = 0$ on B, and let $T: X \to C(B)$ be defined by

$$(Ta)(x) = (1 + a^2)\, e^{-a^2 x}, \quad a \in \mathbb{R}.$$

Then because of

$$\max_{x \in B} |(Ta)(x)| = 1 + a^2$$

$C_\alpha(A)$ is bounded for every $\alpha > 1$ and $\inf_{a \in \mathbb{R}} \|Ta\|_B = 1$ is uniquely achieved for $a = 0$. If we define now $B_k = [\varepsilon_k, 1]$ where $\{\varepsilon_k\}_{k \geq 0}$ is a non-increasing sequence of positive reals which converges to zero, then we have

$$\max_{x \in B_k} |(Ta)(x)| = (1 + a^2)\, e^{-a^2 \varepsilon_k}$$

and none of the $C_\alpha^k(A)$ is bounded. Besides $\inf_{a \in \mathbb{R}} \|Ta\|_{B_k} = 0$ is not attained for any $a \in \mathbb{R}$.

Remark 3. So far we have considered the case that the range of T is in $C(B)$. However, we often will encounter the situation that $T = (T_1, ..., T_q)'$ where T_i maps A into $C(B^{(i)})$ and the $B^{(i)}$, $i \in \{1, 2, ..., q\}$, are compact subsets of $\mathbb{R}^s$. In this case T is a mapping from A into the product space $C(B^{(1)}) \times \cdots \times C(B^{(q)})$ which we equip here with the norm

$$|||r|||_B = \max_{1 \leq i \leq q} \|r_i\|_{B^{(i)}}, \qquad r_i \in C(B^{(i)}),$$

where $\|\cdot\|_{B^{(i)}}$ is the sup-norm on $B^{(i)}$. for every $i \in \{1, ..., q\}$ let now $\{B_k^{(i)}\}_{k \in \mathbb{N}}$ be a sequence of sets which fulfills Assumption 1 and let

$$|||r|||_{B_k} = \max_{1 \leq i \leq q} \|r_i\|_{B_k^{(i)}}, \quad k \in \mathbb{N}.$$

Further, let here $r_i \in C(B^{(i)})$, $i \in \{1, ..., q\}$, be given and let us consider (P) and (P_k) with the two-bar norms being replaced by the three-bar norms. Then if we substitute $|||\cdot|||_B$ and $|||\cdot|||_{B_k}$ for $\|\cdot\|_B$ and $\|\cdot\|_{B_k}$ throughout the preceding part of this paper, resp., and if the properties which we assume above for T are required for all components T_i, $i \in \{1, ..., q\}$, of T here, Theorems 1 and 2 as well as Lemmas 1, 2, and 3 remain valid. In this way convergence of the discretization errors can for instance be shown for the multi-dimensional constrained approximation problems in [5].

4.2. *The case* $1 \leq p < \infty$

Let us demonstrate first that a similar situation as in Example 1 for $p = \infty$ can appear if $1 \leq p < \infty$.

EXAMPLE 2. Let T be defined as in Example 1, but choose $p=1$ here. Then for all $a \neq 0$ we compute

$$\|Ta\|_B = \int_0^1 (1+a^2)\, e^{-a^2 x}\, dx = (1+a^{-2})(1-e^{-a^2}) > 1.$$

Hence $\inf_{a \in \mathbb{R}} \|Ta\|_B = 1$ is uniquely achieved for $a=0$. If now $0 < \xi_1^{(k)} < \xi_2^{(k)} < \cdots < \xi_k^{(k)} \leqslant 1$, we get

$$\begin{aligned} k(1+a^2)\, e^{-a^2 \xi_1^{(k)}} \max_{1 \leqslant i \leqslant k} h_i^{(k)} \geqslant \sum_{i=1}^{k} |(Ta)(\xi_i^{(k)})|\, h_i^{(k)} &= \|Ta\|_{B_k} \\ &\geqslant k(1+a^2)\, e^{-a^2 \xi_k^{(k)}} \min_{1 \leqslant i \leqslant k} h_i^{(k)} \\ &\geqslant k(1+a^2)\, e^{-a^2} \min_{1 \leqslant i \leqslant k} h_i^{(k)}. \end{aligned}$$

Hence for fixed k, $\|Ta\|_{B_k}$ tends to zero for $|a| \to \infty$. Therefore, none of the $C_\alpha^k(A)$ is bounded as well as $\inf_{a \in \mathbb{R}} \|Ta\|_{B_k} = 0$ is not achieved for any $a \in \mathbb{R}$.

However, opposite to the case $p=\infty$, for $1 \leqslant p < \infty$ it is also possible as the following example shows that all of the $C_\alpha^k(A_k)$ are bounded (which implies the existence of solutions to the discrete problems) while $C_\alpha(A)$ is unbounded for every $\alpha > 0$ (and the uniform L_p-problem has no solution).

EXAMPLE 3. We assume again $A = X = \mathbb{R}$, $B = [0, 1]$ and $p=1$. In addition we define $r=0$ on B and $T\colon X \to C(B)$ by

$$(Ta)(x) = (1+a^2)/(1+a^4 x), \quad a \in \mathbb{R}.$$

Then for $a \neq 0$ we have

$$\|Ta\|_B = \int_0^1 \frac{1+a^2}{1+a^4 x}\, dx = \frac{1+a^2}{a^4} \log(1+a^4)$$

which tends to zero for $|a| \to \infty$. Hence $\inf_{a \in \mathbb{R}} \|Ta\|_B = 0$ where the infimum is not achieved for any $a \in \mathbb{R}$. Further, if $0 = \xi_1^{(k)} < \xi_2^{(k)} < \cdots < \xi_k^{(k)} \leqslant 1$, we obtain

$$\|Ta\|_{B_k} = \sum_{i=1}^{k} |(Ta)(\xi_i^{(k)})|\, h_i^{(k)} \geqslant h_1^{(k)}(1+a^2) \geqslant h_1^{(k)} a^2.$$

Consequently, for fixed k, $\|Ta\|_{B_k}$ tends to infinity for $|a| \to \infty$. Therefore, all $C_\alpha^k(A)$ are bounded and all discrete problems possess solutions. It is not seen here whether the ρ_k converge to ρ or not.

Thus for $1 \leqslant p < \infty$ a result corresponding to Lemma 3 cannot be established. However, we can obtain some insight into the situation here if we relate the level sets $C_\alpha(A)$ and $C_\alpha^k(A_k)$ where $C(B)$ is supplied with an L_p-norm, $1 \leqslant p < \infty$, to the corresponding sets where $C(B)$ is associated with the supremum norm. For that matter we write here $\|\cdot\|_{B,p}$ and $\|\cdot\|_{B_k,p}$ instead of $\|\cdot\|_B$ and $\|\cdot\|_{B_k}$ and rename $C_\alpha(S)$ and $C_\alpha^k(S)$ by $C_\alpha^p(S)$ and $C_\alpha^{k,p}(S)$ in order to mark that the (semi-) norm on $C(B)$ equals $\|\cdot\|_{B,p}$ and $\|\cdot\|_{B_k,p}$, resp. Then we can state the following lemma.

LEMMA 4. *Let T be defined on $S \subseteq X$ and $1 \leqslant p < \infty$. Then we have*

(i) $C_\alpha^1(S) \supseteq C^p_{\alpha/(d-c)^{1-1/p}}(S) \supseteq C^\infty_{\alpha/(d-c)}(S)$,

(ii) $C_\alpha^{k,1}(S) \supseteq C^{k,p}_{\alpha/(d-c)^{1-1/p}}(S) \supseteq C^{k,\infty}_{\alpha/(d-c)}(S)$,

(iii) $C_\alpha^{k,\infty}(S) \supseteq C^{k,1}_{\alpha\mu^{(k)}}(S) \supseteq C^{k,p}_{\alpha\mu^{(k)}/(d-c)^{1-1/p}}(S)$

where $\mu^{(k)} = \min_{1 \leqslant i \leqslant k} h_i^{(k)}$.

Proof. If $v \in C[c, d]$ we have by Hölder's inequality

$$\|v\|_{B,1} = \int_c^d |v(x)\cdot 1|\,dx \leqslant \|v\|_{B,p}\,\|1\|_{B,p/(p-1)}$$
$$= \|v\|_{B,p}(d-c)^{1-1/p} \leqslant \|v\|_{B,\infty}(d-c)$$

which yields (i). Similarly we obtain

$$\|v\|_{B_{k,1}} = \sum_{i=1}^{k} |v(\xi_i^{(k)})\,h_i^{(k)1/p}h_i^{(k)(p-1)/p}|$$
$$\leqslant \|v\|_{B_{k,p}}\left\{\sum_{i=1}^{k} h_i^{(k)}\right\}^{(p-1)/p}$$
$$= \|v\|_{B_{k,p}}(d-c)^{1-1/p} \leqslant \|v\|_{B_{k,\infty}}(d-c) \tag{19}$$

and hence we can derive (ii). Finally, with $|v(\xi_r^{(k)})| = \max_{1 \leqslant i \leqslant k}|v(\xi_i^{(k)})|$ we arrive at

$$\|v\|_{B_{k,\infty}}\cdot\mu^{(k)} = |v(\xi_r^{(k)})|\mu^{(k)} \leqslant \sum_{i=1}^{k}|v(\xi_i^{(k)})|\,h_i^{(k)} = \|v\|_{B_{k,1}}$$

which together with (19) implies (iii).

Thus, by previous arguments (cf. Lemma 3) we can draw the following conclusions from Lemma 4.

Conclusions. (i) If T is continuous on A for $\|\cdot\|_{B,\infty}$ and $C_{\hat\alpha}^1(A)$ is bounded for $\hat\alpha = \alpha_0\max(1, d-c)$ with α_0 (5), then (P) has a solution $\hat a \in A$ for all p, $1 \leqslant p \leqslant \infty$.

(ii) Let Assumptions 1 and 2 be fulfilled. Further, let (17) hold true and T be continuous on $A_{\hat{k}}$ for $\hat{k} \in \mathbb{N}$ with respect to $\|\cdot\|_{B,\infty}$. If there is a number $k_0 \geqslant \hat{k}$ such that $C_\alpha^{k_0,\infty}(A_{k_0})$ is bounded for all $\alpha \geqslant 0$, then (P_k) possesses a solution $\hat{a}_k \in A_k$ for each p, $1 \leqslant p \leqslant \infty$, and each $k \geqslant k_0$.

4.3. *Examples*

In this section we want to give two applications of the above theory. For that let $p = \infty$ and $\{B_k\}_{k \in \mathbb{N}}$ be a sequence which in addition to Assumption 1 fulfills (17).

4.3.1 *Generalized Rational Approximation*

Let X be the product space $U \times V$ where U and V are generated by the linearly independent functions $u_1, ..., u_r \in C^1(B)$ and $v_1, ..., v_t \in C^1(B)$, resp., and let

$$\|(u, v)\|_X = \max\{\|u\|_B, \|\dot{u}\|_B, \|v\|_B, \|\dot{v}\|_B\}.$$

Further, for $\delta \in (0, 1)$ given let

$$A = \{(u, v) \in X \mid \delta \leqslant v(x) \leqslant 1, \qquad x \in B\} \tag{20}$$

and correspondingly let

$$A_k = \{(u, v) \in X \mid \delta \leqslant v(x) \leqslant 1, \qquad x \in B_k\}. \tag{21}$$

If we define $T: A \to C(B)$ through

$$[T(u, v)](x) = u(x)/v(x), \qquad x \in B,$$

then (P) becomes a problem of generalized rational approximation. In the following we want to show that for this problem all assumptions of Theorem 1 are fulfilled and that thereby the results of [11] follow immediately (provided that we have $U, V \subset C^1(B)$ instead of $U, V \subset C(B)$). For that let in particular $u_1, ..., u_r$ and $v_1, ..., v_t$ be linearly independent on B_k, $k \in \mathbb{N}$.

LEMMA 5. *There exists a constant $C \in \mathbb{R}$ so that*

$$\max\{\|v\|_B, \|\dot{v}\|_B\} \leqslant C \textit{ for all } v \in V \textit{ with } (u, v) \in A_k, \qquad k \in \mathbb{N}.$$

Proof. Let $v = \sum_{i=1}^t a_i v_i$ with $(u, v) \in A_k$, $k \in \mathbb{N}$. Then we have $\|\sum_{i=1}^t a_i v_i\|_{B_k} \leqslant 1$ which implies $\|a\|_1 \leqslant C_k$ for a constant C_k and all $k \in \mathbb{N}$ (e.g. use Example 2.1 in [16] and note that $C_{k+1} \leqslant C_k$).

LEMMA 6. *There are numbers* $k_3 \in \mathbb{N}$ *and* $\mu_k \in (0, \delta)$, $k \in \mathbb{N}$, *with* $\mu_{k+1} \leqslant \mu_k$ *and* $\lim_{k \to \infty} \mu_k = 0$ *such that in case* $k \geqslant k_3$ A_k *is contained in*

$$\{(u, v) \in X | \delta - \mu_k \leqslant v(x) \leqslant 1 + \mu_k, \qquad x \in B\}.$$

Proof. From Lemma 5 we have for all $k \in \mathbb{N}$ that to each $x \in B$ there is an element $x_k \in B_k$ with

$$|v(x) - v(x_k)| \leqslant C\, h(B_k, B), \qquad v \in A_k.$$

The remainder of the proof follows from Assumption 1 and (17).

LEMMA 7. *There is a* $k_4 \in \mathbb{N}$ *so that* T *can be defined on* A_{k_4} *and* $C_{\alpha_0}^{k_4}(A_{k_4})$ *is bounded.*

Proof. By Lemma 6 T is defined on A_{k_3}. For $(u, v) \in C_{\alpha_0}^{k_3}(A_{k_3})$ we have

$$\alpha_0 \geqslant \|u/v\|_{B_k} \geqslant \|u\|_{B_k}.$$

Therefore, we can conclude for all sufficiently large k that there is a constant C_1 (independent of k) such that further $\max\{\|u\|_B, \|\dot{u}\|_B\} \leqslant C_1$. This together with Lemma 5 yields the requested result.

From Lemma 6 we can easily conclude now that (20), (21) satisfy Assumption 2. Furthermore, it is easily seen that T is continuous on A_{k_4} and that, therefore, Lemma 3 applies here. Then the equicontinuity of the $T(\hat{u}_k, \hat{v}_k)$ on B, where $(\hat{u}_k, \hat{v}_k)$ is a best approximation of (P_k), is a consequence of Lemma 3 (iii).

4.3.2. *Best Approximate Solutions of a Boundary Value Problem*

In [7, 8] Henry considers best approximate polynomial solutions to the following boundary value problem: Find $y \in C^2[0, c]$ so that

$$Ty := y'' + F(x, y, y') + G(x, y, y') = r(x) \text{ on } [0, c] \tag{22}$$

$$y(0) = a_0, \; y'(0) = a_1$$

where F, G, and r satisfy certain assumptions (cf. [8]). We define here X to be the space of all polynomials on $B = [0, c]$ with degree at most $n - 1$ and equip X with the norm

$$\|y\|_X = \max\{\|y\|_B, \|y'\|_B, \|y''\|_B\}, \; y \in X.$$

Futher, we set

$$A = A_k = \{y \in X | \, y(0) = a_0, \, y'(0) = a_1\}.$$

Finally, we require that B_1 contains at least $n + l$ points (for l see [8]). Obviously T maps A continuously into $C[0, c]$.

In [7, 8] the existence of solutions of (P) and (P_k) as well as the convergence of the ρ_k to ρ has been established. With the following we show that the assumptions made in [8] immediately imply the boundedness of the $C_\alpha^k(A_k)$ for all $k \in \mathbb{N}$ so that the requested existence of solutions and the convergence of the discretization errors follow from Lemma 3 and Theorem 1.

LEMMA 8. *For every $\alpha \geqslant 0$ and every $k \in \mathbb{N}$, $C_\alpha^k(A_k)$ is bounded.*

Proof. Let $v_i = x^i$, $0 \leqslant i \leqslant n-1$, $\beta = (\beta_0, ..., \beta_{n-1}) \in \mathbb{R}^n$, and $a(x) = \sum_{i=0}^{n-1} \beta_i v_i(x)$. Hence for $a \in C_\alpha^k(A_k)$ we have $\|Ta\|_{B_k} \leqslant \alpha$ so that from (22) we obtain

$$\|G(\cdot, a, a')\|_{B_k} \leqslant \alpha + \|a''\|_{B_k} + \|F(\cdot, a, a')\|_{B_k}. \tag{23}$$

If we define now

$$\sigma_k = \min_{\|\beta\|_1 = 1} \left\| u\, \Phi\left(\sum_{i=0}^{n-1} \frac{\beta_i}{\|\beta\|_1} v_i, \sum_{i=0}^{n-1} \frac{\beta_i}{\|\beta\|_1} v_i' \right) \right\|_{B_k},$$

we conclude from the assumptions in [8] that σ_k is positive (cf. Lemma 1, [8]) and that further (23) implies

$$\|\beta\|_1^\gamma \sigma_k \leqslant \alpha + \|\beta\|_1 \max_{0 \leqslant i \leqslant n-1} \|v_i''\|_B + O(\|\beta\|_1^\eta)$$

(where $\alpha := \gamma$ and $r := \|\beta\|_1$ here). Because of $\gamma > \max(1, \eta)$ there exists a constant $C > 0$ so that $\|\beta\|_1 \leqslant C$.

Hence all assumptions of Lemma 3 are fulfilled here. The equicontinuity of the $\{T\hat{a}_k\}_{k \in \mathbb{N}}$ finally can be established by Lemma 3(iii) and by the continuity assumptions on F, G, and r (cf. [8], p. 261).

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